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Eighteenth Order Convergent Method for Solving Non-Linear Equations

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ABSTRACT*:*

In this paper, we suggest and discuss an iterative method for solving nonlinear equations of the type f(x) = 0 having eighteenth order convergence. This new technique based on Newton's method and extrapolated Newton's method. This method is compared with the existing ones through some numerical examples to exhibit its superiority.
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INTRODUCTION

We consider finding the zero's of a nonlinear equation

$$f(x)=0 \qquad \text{...(1.1)}$$

Where $f : D \subset R \to R$ is a scalar function on an open interval *D* and f(x) may be algebraic, transcendental or combined of both. The most widely used algorithm for solving (1.1) by the use of value of the function and its derivative is the well known quadratic convergent Newton's method (NM) given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{...(1.2)}$$

(n=0,1,2,...)

starting with an initial guess x_0 which is in the vicinity of the exact root x*. The efficiency index of Newton's method is $\sqrt[2]{2} = 1.4142$.

The Extrapolated Newton's method (ENM) suggested by V.B.Kumar, Vatti *et.al* [11] which is developed by extrapolating Newton's method (1.2) introducing a parameter 'α_n' given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left[x_n - \frac{f(x_n)}{f'(x_n)}\right]$$

$$\Rightarrow x_{n+1} = x_n - \alpha_n \frac{f(x_n)}{f'(x_n)} \qquad \text{...(1.3)}$$

(n=0,1,2,...)

Here, the optimal choice for the parameter 'α_n' is

$$\alpha_n = \frac{2}{2-\rho_n} \quad ...(1.4)$$

Where $\rho_n = \frac{f(x_n)f''(x_n)}{\left[f'(x_n)\right]^2}$ ...(1.5)

...

Combining (1.3), (1.4) and (1.5), one can have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \cdot \frac{1}{\left[1 - \frac{f(x_n)f''(x_n)}{2\left[f'(x_n)\right]^2}\right]} \quad ...(1.6)$$

(n=0,1,2,...)

Which is same as Halley's method having third order convergence requires three functional evaluations. The efficiency index of this method is $\sqrt[3]{3} = 1.4422$.

A three step Predictor-corrector Newton's Halley method (PCNH) suggested by Mohammed and Hafiz [10] (see[1 to 10]), is given by:

For a given x_0, we compute x_{n+1} by using

$$\left.\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ y_n &= w_n - \frac{2f(w_n)f'(w_n)}{2\left[f'(w_n)\right]^2 - f(w_n)f''(w_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{\left[f(y_n)\right]^2 f''(y_n)}{2\left[f'(y_n)\right]^3} \\ & \qquad (n = 0,1,2,\ldots) \end{aligned}\right\} \quad ...(1.7)$$

This method has eighteenth order convergence and its efficiency index is

$$\sqrt[8]{18} = 1.4352$$

In section 2, we develop and discuss a three step iterative method and the convergence criteria is discussed in section 3. Few numerical examples are considered to show the superiority of this method in the concluding section.

Eighteenth Order Convergent Method (Eocm)

Consider x* be the exact root of (1.1) in an open interval *D* in which f(x) is continuous and has well defined first and second derivatives. Let x_n be the n^{th} approximate to the exact root x* of (1.1) and

$$x^* = x_n + e_n \quad ...(2.1)$$

where is the error at the n^{th} stage.

Therefore, we have

$$f\left(x^*\right) = 0 \quad ...(2.2)$$

Expanding by Taylor's series about, we have

$$f\left(x^*\right) = f(x_n) + \left(x^* - x_n\right)f'(x_n) + \frac{\left(x^* - x_n\right)^2}{2!} f''(x_n) + \ldots \quad ...(2.3)$$

$$f\left(x^*\right) = f(x_n) + e_n f'(x_n) + \frac{e_n^2}{2} f''(x_n) + \ldots \quad ...(2.4)$$

Assuming e_n is small enough and neglecting higher powers of e_n starting from e_n^3 onwards, we obtain from (2.2) and (2.4) as

$$e_n^2 f''(x_n) + 2e_n f'(x_n) + 2f(x_n) = 0$$

$$\Rightarrow e_n = \left[-2f'(x_n) \pm \sqrt{4f'(x_n) - 8f(x_n)f''(x_n)}\right] \Big/ 2f''(x_n)$$

Where $\hat{\rho} = \frac{f(x_n)f''(x_n)}{\left[f'(x_n)\right]^2}$ ...(2.5)

Now, $e_n = -\frac{f'(x_n)}{f''(x_n)} \frac{\left[1 - \sqrt{1-2\hat{\rho}}\right]\left[1 + \sqrt{1-2\hat{\rho}}\right]}{\left[1 + \sqrt{1-2\hat{\rho}}\right]}$

i.e., $e_n = -\frac{2f(x_n)}{f'(x_n)}\left[\frac{1}{1+\sqrt{1-2\hat{\rho}}}\right]$ ...(2.6)

Replacing x^* by x_{n+1} in (2.1) and from (2.5) &(2.6), we obtain

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)} \cdot \frac{1}{\left[1 + \sqrt{1 - 2\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}\right]} \quad ...(2.7)$$

This scheme (2.7) allows us to propose the following algorithm with the method (1.2) as the first step and the method (1.6) as the second step.

Algorithm 2.1: For a given x_0, compute x_{n+1} by the iterative schemes

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad ...(2.8)$$

$$y_n = w_n - \frac{f(w_n)}{f'(w_n)}\left[\frac{1}{1-\hat{\rho}/2}\right] \quad ...(2.9)$$

Where $\hat{\rho}_n = \dfrac{f(w_n)f''(w_n)}{[f'(w_n)]^2}$ ...(2.10)

$$x_{n+1} = y_n - \frac{2f(y_n)}{f'(y_n)}\left[\frac{1}{1+\sqrt{1-2\rho_n}}\right] \quad ...(2.11)$$

(n=0,1,2,...)

Where $\rho_n = \dfrac{f(y_n)f''(y_n)}{[f'(y_n)]^2}$ ...(2.12)

This algorithm (2.1) requires 3 functional evaluations, 3 of its first derivatives and 2 of its second derivatives and can be called as Eighteenth order Convergent Method (EOCM).

Convergence Criteria

Theorem 3.1. Let $x^* \in D$ be a single zero of a sufficiently differentiable function $f : D \subset R \to R$ for an open interval *D*. If x_0 is in the vicinity of x*, then the algorithm (2.1) has eighteenth order convergence.

Proof: Let x* be a single zero of (1.1) and

$$x^* = x_n + e_n \quad ...(3.1)$$

Then, $f(x^*) = 0$ ...(3.2)

If x_n be the n[th] approximate to the root of (1.1), then expanding $f(x_n)$ about x* using Taylor's expansion, we have

$$f(x_n) = f(x^*) + e_n f'(x^*) + \frac{e_n^2}{2!}f''(x^*) + \frac{e_n^3}{3!}f'''(x^*) + \ldots f(x_n)$$

$$= f'(x^*)\left[e_n + \frac{1}{2!}\frac{f''(x^*)}{f'(x^*)}e_n^2 + \frac{1}{3!}\frac{f'''(x^*)}{f'(x^*)}e_n^3 + \frac{1}{4!}\frac{f^{iv}(x^*)}{f'(x^*)}e_n^4 + \ldots\right] f(x_n)$$

$$= f'(x^*)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \ldots\right] \quad ...(3.3)$$

Where

$$c_j = \frac{1}{j!} \cdot \frac{f^j(x^*)}{f'(x^*)}, \quad (j = 2, 3, 4, \ldots)$$

And,

$$f'(x_n) = f'(x^*)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \ldots\right] \quad ...(3.4)$$

Now,

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 - (2c_3 - 2c_2^2)e_n^3 - (3c_4 - 7c_2 c_3 + 4c_2^3)e_n^4 + o(e_n^5) \quad ...(3.5)$$

From (2.8), (3.1) and (3.5), we have

$$w_n = x^* + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 4c_2^3)e_n^4 + o(e_n^5) \quad ...(3.6)$$

Let $W = w_n - x^*$ ...(3.7)

Then, expanding $f(w_n),\ f'(w_n),\ f''(w_n)$ about X* by using (3.6), we obtain

$$f(w_n) = f(x^*) + (w_n - x^*) f'(x^*) + \frac{(w_n - x^*)^2}{2!} f''(x^*) + \frac{(w_n - x^*)^3}{3!} f'''(x^*) + \ldots \quad \ldots(3.8)$$

$$= f'(x^*)\left[W + c_2W^2 + c_3W^3 + c_4W^4 + \ldots\right] \quad \ldots(3.9)$$

$$f'(w_n) = f'(x^*) + (w_n - x^*) f''(x^*) + \frac{(w_n - x^*)^2}{2!} f'''(x^*) + \ldots \quad \ldots(3.10)$$

Combining (3.6) to (3.10) as done in [10] and from (2.9), we obtain

$$y_n = x^* + T \quad \ldots(3.11)$$

where $T = (c_2{}^2 - c_3)W^3$ ...(3.12)

Now, expanding $f(y_n),\ f'(y_n),\ f''(y_n)$ about X* by using (3.11), we obtain

$$f(y_n) = f'(x^*)\left[T + c_2T^2 + c_3T^3 + c_4T^4 + \ldots\right] \quad \ldots(3.13)$$

$$f'(y_n) = f'(x^*)\left[1 + 2c_2T + 3c_3T^2 + 4c_4T^3 + 5c_5T^4 + \ldots\right] \quad \ldots(3.14)$$

$$f''(y_n) = f'(x^*)\begin{bmatrix} 2c_2 + 6c_3T + 12c_4T^2 \\ +20c_5T^3 + 30c_6T^4 + \ldots \end{bmatrix} \quad \ldots(3.15)$$

and $\rho_n = \dfrac{f(y_n) f''(y_n)}{\left[f'(y_n)\right]^2}$

$$= \frac{2c_2T + (6c_3 + 2c_2{}^2)T^2 + (12c_4 + 8c_2c_3)T^3 + (20c_5 + 14c_2c_4 + 6c_3{}^2)T^4 + \ldots}{1 - 4c_2T + (12c_2{}^2 - 6c_3)T^2 + (36c_2c_3 - 32c_2{}^3 - 8c_4)T^3 + (36c_2c_3 - 32c_2{}^3 - 8c_4)T^3 + (80c_2{}^4 - 144c_2{}^2c_3 + 48c_2c_4 + 27c_3{}^2 - 10c_5)T^4 + \ldots}$$

$$= P_1T + P_2T^2 + P_3T^3 + P_4T^4 + \ldots \quad \ldots(3.18)$$

where, $P_1 = 2c_2,\ P_2 = -6c_2{}^2 + 6c_3,$

$P_3 = 16c_2{}^3 - 28c_2c_3 + 12c_4,$

$P_4 = 100c_2{}^2c_3 - 40c_2{}^4 - 50c_2c_4 - 30c_3{}^2 + 20c_5$

Now,

$$\sqrt{1 - 2\rho_n} = 1 - P_1T + \left(-\frac{P_1{}^2}{2} - P_2\right)T^2 + \left(-P_3 - P_1P_2 - \frac{P_1{}^3}{2}\right)T^3 + \left(-P_4 - \frac{P_2{}^2}{2} - P_1P_3 - \frac{3}{2}P_1{}^2P_2 - \frac{5}{8}P_1{}^4\right)T^4 + \ldots \quad \ldots(3.17)$$

and,

$$1 + \sqrt{1 - 2\rho_n} = 2\begin{bmatrix} 1 - \frac{P_1}{2}T - \left(\frac{P_1{}^2}{4} + \frac{P_2}{2}\right)T^2 - \left(\frac{P_3}{2} + \frac{P_1P_2}{2} + \frac{P_1{}^3}{4}\right)T^3 \\ -\left(\frac{P_4}{2} + \frac{P_2{}^2}{4} + \frac{P_1P_3}{2} + \frac{3}{4}P_1{}^2P_2 + \frac{5}{16}P_1{}^4\right)T^4 + \ldots \end{bmatrix}$$

$$1 + \sqrt{1 - 2\rho_n} = 2\left[1 + M_1T + M_2T^2 + M_3T^3 + M_4T^4 + \ldots\right] \quad \ldots(3.18)$$

where, $M_1 = -c_2,\ M_2 = 2c_2{}^2 - 3c_3,$

$M_3 = -4c_2{}^3 + 8c_2c_3 - 6c_4,$

$M_4 = -4c_2{}^2c_3 + 4c_2{}^4 + 13c_2c_4 - 3c_3{}^2 - 10c_5$

Now again,

$$\left[1 + \sqrt{1 - 2\rho_n}\right]^{-1} = \frac{1}{2}\begin{bmatrix} 1 - M_1T + (M_1{}^2 - M_2)T^2 + (2M_1M_2 - M_3 - M_1{}^3)T^3 \\ + (M_2{}^2 + 2M_1M_4 - M_4 - 3M_1{}^2M_2 + M_1{}^4)T^4 + \ldots \end{bmatrix}$$

$$\left[1 + \sqrt{1 - 2\rho_n}\right]^{-1} = \frac{1}{2}\left[1 + N_1T + N_2T^2 + N_3T^3 + N_4T^4 + \ldots\right] \quad \ldots(3.19)$$

Where, $N_1 = c_2,\ N_2 = -c_2{}^2 + 3c_3,$

$N_3 = c_2{}^3 - 2c_2c_3 + 6c_4,$

$N_4 = -15c_2{}^2c_3 + 3c_2{}^4 - c_2c_4 + 12c_3{}^2 + 10c_5$

From (3.13) and (3.14), we obtain

$$\frac{f(y_n)}{f'(y_n)} = T - c_2T^2 + (2c_2{}^2 - 2c_3)T^3 + (7c_2c_3 - 4c_2{}^3 - 3c_4)T_n{}^4 + o(T^5) \quad \ldots(3.20)$$

From (3.19) and (3.20), we obtain

$$\frac{f(y_n)}{f'(y_n)}\cdot\left[1+\sqrt{1-2\hat{\rho}_n}\right]^{-1}=T+\left[N_1-c_2\right]T^2$$
$$+\left[N_2-N_1c_2+2c_2^{\,2}-2c_3\right]T^3$$
$$+\left[\begin{array}{l}N_3-N_2c_2+N_1\left(2c_2^{\,2}-2c_3\right)+\\7c_2c_3-4c_2^{\,3}-3c_4\end{array}\right]T_n^{\,4}+o\left(T^5\right)$$

$$=T+[0]T^2+\left[c_3\right]T^3+\left[\begin{array}{l}N_3-N_2c_2+N_1\left(2c_2^{\,2}-2c_3\right)\\+7c_2c_3-4c_2^{\,3}-3c_4\end{array}\right]T_n^{\,4}+o\left(T^5\right) \quad ...(3.21)$$

Combining (3.13) to (3.21) and from (2.11), we obtain

$$x_{n+1}=x^*+T-\left[T+c_3T^3+\ldots\right]$$
$$=x^*-c_3T^3+\ldots$$
$$=x^*-c_3\left[\left(c_2^{\,2}-c_3\right)W^3\right]^3+\ldots$$

i.e., $e_{n+1}=c_3\left(c_2^{\,2}-c_3\right)^3W^9+\ldots$

$$\Rightarrow e_{n+1}=c_3\left(c_2^{\,2}-c_3\right)^3\left[c_2e_n^{\,2}+\left(2c_3-2c_2^{\,2}\right)e_n^{\,3}\right]^9+\ldots$$

$$\Rightarrow e_{n+1}=c_3c_2^{\,9}\left(c_2^{\,2}-c_3\right)^3e_n^{\,18}+o\left(e_n^{\,19}\right) \quad ...(3.22)$$

Hence, this method has eighteenth order convergence and its efficiency index is $\sqrt[8]{18}=1.4352$.

Case 3.1: By expanding $\sqrt{1-2\rho_n}$ appearing in the denominator of the third step of algorithm 2.1, we obtain

$$x_{n+1}=y_n-\frac{2f(y_n)}{f'(y_n)}\cdot\frac{1}{\left[2-\rho_n-\frac{\rho_n^{\,2}}{2}-\frac{\rho_n^{\,3}}{2}\right]}$$

$$\Rightarrow x_{n+1}=y_n-\frac{f(y_n)}{f'(y_n)}\left[1+\frac{\rho_n}{2}+\frac{\rho_n^{\,2}}{2}+\frac{5\rho_n^{\,3}}{8}\right] \quad ...(3.23)$$

where ρ_n is as given in (2.12).

Considering the first degree and second degree terms of the expression lying within the square brackets of the formula (3.23) and from the algorithm 2.1, we have the following two more algorithms.

Algorithm 3.1: For a given, compute by the iterative schemes

$$w_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$y_n=w_n-\frac{f(w_n)}{f'(w_n)}\left[\frac{1}{1-\rho_n/2}\right]$$

Where $\rho_n=\dfrac{f(w_n)f''(w_n)}{\left[f'(w_n)\right]^2}$

$$x_{n+1}=y_n-\frac{f(y_n)}{f'(y_n)}-\frac{\left[f(y_n)\right]^2f''(y_n)}{2\left[f'(y_n)\right]^3}$$

(n=0,1,2,...)

which is same as the eighteenth order convergent method proposed by Mohammed and Hafiz [10].

Algorithm 3.2: For a given x_0, compute x_{n+1} by the iterative schemes

$$w_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$y_n=w_n-\frac{f(w_n)}{f'(w_n)}\left[\frac{1}{1-\rho_n/2}\right]$$

Where $\hat{\rho}_n=\dfrac{f(w_n)f''(w_n)}{\left[f'(w_n)\right]^2}$

$$x_{n+1}=y_n-\frac{f(y_n)}{f'(y_n)}-\frac{\left[f(y_n)\right]^2f''(y_n)}{2\left[f'(y_n)\right]^3}-\frac{\left[f(y_n)\right]^3\left[f''(y_n)\right]^2}{2\left[f'(y_n)\right]^5}$$

As done in convergence criteria above in section 3, one can easily obtain.Therefore, the algorithm (3.2) has eighteenth order convergence.

Numerical Examples

We consider the same examples considered by Mohammed and Hafiz[10] and compared EOCM with NM and PCNH methods. The computations are carried out by using mpmath-PYTHON software programming and comparison of number of iterations for these methods are obtained such that $\left|x_{n+1}-x_n\right|<10^{-201}$ and $\left|f\left(x_{n+1}\right)\right|<10^{-201}$.

Table 1: Comparison of different methods

Method	Initial Guess x_0	The equation $f(x)=0$ and its root by respective methods	No. of iterations	$\left\|x_{n+1}-x_n\right\|$	$\left\|f\left(x_{n+1}\right)\right\|$
	1	$f(x)=x^3+4x^2-10$			
NM		1.365230013414096845760806828981666078331164746 7	10	3.43E-200	-5.75E-199
PCNH		1.3652300134140968457608068289816660783311647467	5	-3.27E-201	-5.75E-199
EOCM		1.3652300134140968457608068289816660783311647467	3	-3.27E-201	1.07E-198
	1.3	$f(x)=\sin^2 x-x^2+1$			
NM		1.4044916482153412260350868177868680771766025759	9	6.86E-200	1.72E-199
PCNH		1.4044916482153412260350868177868680771766025759	4	-6.53E-201	-7.67E-200
EOCM		1.4044916482153412260350868177868680771766025759	3	9.31E-200	-7.67E-200
	2	$f(x)=x^2-e^x-3x+2$			
NM		0.25753028543986076045536730493724178138453699 34	9	8.2E-202	-3.2E-201
PCNH		0.2575302854398607604553673049372417813845369934	4	2.45E-201	-3.27E-201
EOCM		0.2575302854398607604553673049372417813845369934	4	2.45E-201	-3.27E-201
	-4	$f(x)=x^2-e^x-3x+2$			
NM		0.2575302854398607604553673049372417813845369934	11	8.2E-202	-3.3E-201
PCNH		0.2575302854398607604553673049372417813845369934	5	2.45E-201	-3.27E-201
EOCM		0.2575302854398607604553673049372417813845369934	4	2.45E-201	-3.27E-201
	1.7	$f(x)=\cos x-x$			
NM		0.7390851332151606416553120876738734040134117589	9	1.63E-201	2.45E-201
PCNH		0.7390851332151606416553120876738734040134117589	4	-4.9E-201	2.45E-201
EOCM		0.7390851332151606416553120876738734040134117589	3	-4.9E-201	2.45E-201
	2.5	$f(x)=(x-1)^3-1$			
NM		2.0	10	0	0
PCNH		2.0	4	0	0
EOCM		2.0	4	0	0
	2	$f(x)=x^3-10$			
NM		2.1544346900318837217592935665193504952593449421	9	1.63E-200	-2.1E-199
PCNH		2.1544346900318837217592935665193504952593449421	4	-4.9E-200	-2.1E-199
EOCM		2.1544346900318837217592935665193504952593449421	3	-4.9E-200	-2.1E-199

The above computational results exhibit the superiority of the new method EOCM over the Newton's method and PCNH method in terms of number of iterations and accuracy.

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