Fixed Point Theorems for Multivalued Mappings Satisfying Functional Inequality

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ABSTRACT

In this paper we have established a fixed point theorem for multivalued mappings and generalized the result of Sayyed, Sayyed and Badshah[6].
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INTRODUCTION

The fixed poing theory for single valued maps is very rich and well developed, the multivalued case is not. Note that multivalued mappings play a major role in many areas as in studying disjunctive logic programs. Nadler[5] was first to extend Banach contraction principle to multivalued contraction mapping.

The first theorem regarding to continuity of Fixed points of contraction mapping was presented by Bonsal[1].

Theorem 1

Let (X, d) be complete metric space and let F and Fn (n = 1, 2,...) be contraction mappings of X into itself with the some Lipschitz constant K < 1, and with fixed points u and u_n respectively. Suppose that $\lim_{n \to \infty} F_n X = Fx$ for every $x \in X$.. Then

U_n=u.

Subsequently Nadler[4] obtained, results concerning sequence of contracting mappings.

$\lim_{n \to \infty}$

Theorem 2

Let (X, d) be a metric space, let $F_1 : X \to X$ be a mapping with atleast one fixed point xi, for each i = 1, 2, and .... let F_o $X \to X$ be a contraction mapping with fixed point x_o. If the sequence F_i converges uniformly to F_o, then sequence $\{x_i\}$ converges to x_o.

Let A be a closed bounded subset of Hilbert space X, d the metric of X and H the Haudorff metric on the closed subset of A generated by d. It is assumed that the family of set valued mappings Fk, (K = 0, 1, 2,. ..) satisfy the following conditions.

1. F_k, (X) is non empty closed convex subset of A for each $x \in A$
2. Each F_k is set valued contraction, that is there is a $\lambda \in [0, 1]$ such that $H(F_k(x), F_k(y)) \le \lambda\, d(x,y)$ for x, $y \in A$ and K = 0, 1, 2, ...
3. $H(F_k(x), F_O(x)) = 0$ uniformly for all $x \in A$

Let CB (X) denote the set of non empty closed bounded subset of X. The following is a simple co sequences of the definition of Haudorff metric H. Let A, B, $\in$ CB (X) and $a \in A$. If n > 0 then there exist $b \in B$ such that $d(a,b) \leq H(A, B) + n$, i.e.., $d(a, b,) \leq PH(A, B)$ Where $P > 1$. It $A, B, \in C(X)$ and $a \in A$, then there exist $b \in B$ such that $d(a,b) \leq H(A, B)$.

The following theorem is were proved by Bose and Mukherjee[2].

Definition 1

Let $H(A, B) = \text{Inf}\{\in / A \subset N(\in, B)$ and $B \subset N(\in, A)\}$ for $A, B, \in CB(X)$ where $N(\in, C) = \{x \in X \mid d(x,c) < \in$ for some $c \in C\}$ where $\in > O$ and $C \in CB(X)$. The function H is a metric on CB (X) and is called Haudorff metric. The metric H depends on the metric d of X and two equivalent metrics on X may not generate equivalent Hausdorff metrics for CB (X) (Kelley) [3])

Definition 2

Let (x_1, d_1) and (y_1, d_2) be two metric spaces. Let $F : (X_1, d_1) \to CB(Y)$. F is said to be multivalued contraction mapping if and only if

$$H(fx, fy) \leq K\, d1(x, y)\ x, y \in X$$

Where $0 \leq K < 1$, is a fixed real number

Definition 3

Let (X, d) be a complete metric space. A mapping $F: X \to X$ is said to be of generalized another type if

$$[d(Fx, Fy)]^2 \leq \alpha\,[d(x, Fx), d(y, Fy) + d(x, Fy)\, d(y, Fx)] + \beta\,[d(x, Fx)\, d(x, Fy) + d(y, Fy)\, d(y, Fx)]$$

where α, β are non negative numbers such that $0 \leq \alpha + \beta < 1$.

Let CB (X) denote the set of non empty closed bounded subset of X.

Theorem 3

Let $\{F_n\}$ be a sequence of self mappings of X having at least one fixed point x_n each and let $\{F_n\}$ converges uniformly to Fo, a mapping of generalized Kannan - Reich type. Let Xo be the unique fixed point of Fo. Then $Xn \to Xo$.

Theorem 4

Let $\{F_n\}$ be a sequence of mapping of generalized Kannan - Riech type and let $\{F_n\}$ converges pointwise to F, a generalized Kannan - Reich type mapping. Let X_n and X_o be fixed points of F_n and F resp. Then $Xn \to Xo$.

The following is a simple consequences of the definition of Hausdorff metric H. Let $A, B \in CB(X)$ and $a \in A$. if $\eta > 0$, then there exist $b \in B$, such that $d(a, b) \leq H(A, B) + \eta$ i.e., $d(a, b) \leq PH(A, B)$ where $p > 1$. If $A, B \in C(X)$ and $a \in A$, then there exist $b \in B$ such that $d(a,b) \leq H(A,B)$.

The following theorem were proved by Bose and Mukherjee[2].

Theorem 5

Let (X, d) be complete bounded metric space and let $F: X \to CL(X)$ be mapping satisfying the following condition.

$$d(F(x),F(y) \leq \alpha\left[\frac{\{d(x,F(x))\}^2+\{d(y,F(y))\}^2}{d(x,F(x))+d(y,F(y))}\right]+\beta d(x,y)$$

where α and β are non-negative numbers $0 < 2\alpha + \beta < 1$. then F has a fixed point.

Proof

Let $x_0 \in X$ consider the sequence $\{x_n\}$ where $X_{2n+1} \in F(X_{2n})$

1. Let us assume $d(F(x_o), F(x_1)) = 0$
   $\Rightarrow$ then f has fixed point.

II. Now if $d(F(x_0), F(x1)) \neq 0$
   Then E as number $h > d[F(x_0), F(x_1)]$
   Such that $d(x_1, x_2) \leq h$
   Let $h = p^{-1} d(F(x_0), F(x_1))$
   Where $P = (a+b)^{1/2}$

Then

$$d(x_1,x_2) \leq p^{-1}\left[\alpha\frac{\{d(x_0,F(x_0))\}^2+\{d(x_1,F(x_1))\}^2}{d(x_0,F(x_0))+d(x_1,F(x_1))}\right]+\beta d(x_0,x_1)$$

$$Pd(x_1,x_2) \leq \alpha\left[\frac{\{d(x_0,x_1)\}^2+\{d(x_1,x_2)\}^2}{d(x_0,x_1)+d(x_1,x_2)}\right]+\beta d(x_0,x_1)$$

$$< \alpha[d(x_0, x_1) + d(x_1, x_2)] + \beta d(x_0, x_1)$$
$$< \alpha[d(x_0, x_1) + d(x_1, x_2)] + \beta d(x_0, x_1)$$
$$Pd(x_1, x_2) < \alpha(\alpha + \beta) d(x_0, x_1) + \alpha d(x_1, x_2)$$
$$d(x_1, x_2) < \left(\frac{\alpha + \beta}{p - \alpha}\right) d(x_0, x_1)$$

continuing in similar fashion there exist $x_3 \in F(x_2)$ such that

$$d(x_2, x_3) < \left(\frac{\alpha + \beta}{p - \alpha}\right)^2 d(x_0, x_1)$$

We have $\alpha > 0$ $a = \left(\frac{\alpha + \beta}{p - \alpha}\right)$

Further,

$$d(x_{2n+1}, x_{2n+2}) < \frac{(\alpha + \beta)^{2n+1}}{(p - \alpha)^{2n+1}} d(x_0, x_1)$$

$$d(x_{2n+1}, x_{2n+2}) < \alpha^{2n+1} d(x_0, x_1)$$

It can be easily seen that the sequence is Cauchy sequence and hence converges to some points $u \in X$ consider.

$d(F(u), u) \leq d(F(u), x_{n+1}) + d(x_{n+1}, u)$

$\leq d(F(u), F(x_n)) + d(x_{n+1}, u)$

$$\leq p^{-1}\left[\frac{\alpha\{d(u, F(u))\}^2 + (x_n, F(x_n))}{d(u, F(u)) + d(x_n, F(x_n))}\right] + \beta d(u, x_n)$$

$Pd(F(u), u) < \alpha\, [d(u), F(u)) + d(x_n, f(x_n))] + \beta\, d(u, x_n) + d(x_{n+1}, u)$

$< \alpha\, [d(u), f(u)) + d(x_n, f(x_{n+1}))] + \beta\, d(u, x_n) + d(xn_{+1}, u)$

$\Rightarrow (P-\alpha)\, d(f(u), u) \leq \alpha d\, (x_n, x_{n+1}) + \beta\, d\, (u, x_n) + d\, (x_{n+1}, u)$

Taking limit $n \to \infty$ them

$$(P-\alpha)\, d\, (F(u), u) = 0$$
$$\Rightarrow F(u) = u$$

u is a fixed point for F.

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