INTRODUCTION

The fixed point theory for single valued maps is very rich and well developed, the multivalued case is not. Note that multivalued mappings play a major role in many areas as in studying disjunctive logic programs. Nadler was first to extend Banach contraction principle to multivalued contraction mapping.

The first theorem regarding to continuity of Fixed points of contraction mapping was presented by Bonsal.

Theorem 1

Let \((X, d)\) be a complete metric space and let \(F\) and \(F_n\) \((n = 1, 2, \ldots)\) be contraction mappings of \(X\) into itself with the same Lipschitz constant \(K < 1\), and with fixed points \(u\) and \(u_n\) respectively. Suppose that \(\lim_{n \to \infty} F_n X = Fx\) for every \(x \in X\). Then

\[ u_n = u. \]

Subsequently Nadler obtained, results concerning sequence of contracting mappings.

Theorem 2

Let \((X, d)\) be a metric space, let \(F_i : X \to X\) be a mapping with at least one fixed point \(x_i\), for each \(i = 1, 2, \ldots\) let \(F_0 : X \to X\) be a contraction mapping with fixed point \(x_0\). If the sequence \(F_i\) converges uniformly to \(F_0\), then sequence \(\{x_i\}\) converges to \(x_0\).

In this paper we have established a fixed point theorem for multivalued mappings and generalized the result of Sayyed, Sayyed and Badshah. AMS 2000 Subject Classifications: Primary 54H25, Secondary 47H10

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ABSTRACT

In this paper we have established a fixed point theorem for multivalued mappings and generalized the result of Sayyed, Sayyed and Badshah.


Fixed Point Theorems for Multivalued Mappings Satisfying Functional Inequality

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Let \(A\) be a closed bounded subset of Hilbert space \(X\), \(d\) the metric of \(X\) and \(H\) the Hausdorff metric on the closed subset of \(A\) generated by \(d\). It is assumed that the family of set valued mappings \(F_k\), \((K = 0, 1, 2, \ldots)\) satisfy the following conditions.

1. \(F_k : A\) is non empty closed convex subset of \(A\) for each \(x \in A\)
2. Each \(F_k\) is set valued contraction, that is there is a \(\lambda \in [0, 1]\) such that \(H(F_k(x), F_k(y)) \leq \lambda d(x,y)\) for \(x, y \in A\) and \(K = 0, 1, 2, \ldots\)
3. \(H(F_k(x), F_0(x)) = 0\) uniformly for all \(x \in A\)

Let \(CB(X)\) denote the set of non empty closed bounded subset of \(X\). The following is a simple co sequences of the definition of Hausdorff metric \(H\). Let \(A, B, \in CB(X)\) and \(a \in A\). If \(n > 0\) then
there exist \( b \in B \) such that \( d(a, b) \leq H(A, B) + n \), i.e., \( d(a, b) \leq PH(A, B) \) where \( P > 1 \). If \( A, B, a \in C(X) \) and \( a \in A \), then there exist \( b \in B \) such that \( d(a, b) \leq H(A, B) \).

The following theorem is were proved by Bose and Mukherjee².

**Definition 1**

Let \( H(A, B) = \inf \{ \epsilon / A \subset N(\epsilon, B) \text{ and } B \subset N(\epsilon, A) \} \) for \( A, B, a \in C(X) \) where \( N(\epsilon, C) = \{ x \in X \mid d(x, c) < \epsilon \} \) for some \( c \in C \) where \( \epsilon > 0 \) and \( C \in CB(X) \). The function \( H \) is a metric on \( CB(X) \) and is called Haudorff metric. The metric \( H \) depends on the metric \( d \) of \( X \) and two equivalent metrics on \( X \) may not generate equivalent Hausdorff metrics for \( CB(X) \) (Kelley) [3]).

**Definition 2**

Let \((x_1, d_1)\) and \((y_1, d_2)\) be two metric spaces. Let \( F : (X_1, d_1) \to CB(Y) \). \( F \) is said to be multivalued contraction mapping if and only if

\[
H(fx, fy) \leq K d_1(x, y) \quad x, y \in X
\]

Where \( 0 \leq K < 1 \), is a fixed real number.

**Definition 3**

Let \((X, d)\) be a complete metric space. A mapping \( F : X \to X \) is said to be of generalized another type if:

\[
[d(Fx, Fy)]^2 \leq \alpha \left( [d(x, Fx) + d(y, Fy)] + \beta [d(x, Fx) + d(y, Fy)] \right) + \beta d(x, y)
\]

where \( \alpha \) and \( \beta \) are non-negative numbers such that \( 0 \leq \alpha + \beta < 1 \).

Let \( CB(X) \) denote the set of non empty closed bounded subset of \( X \).

**Theorem 3**

Let \( \{F_n\} \) be a sequence of self mappings of \( X \) having at least one fixed point \( x_n \) each and let \( \{F_n\} \) converges uniformly to \( F_0 \), a mapping of generalized Kannan - Reich type. Let \( X_0 \) be the unique fixed point of \( F_0 \). Then \( X_n \to X_0 \).

**Theorem 4**

Let \( \{F_n\} \) be a sequence of mapping of generalized Kannan - Reich type and let \( \{F_n\} \) converges pointwise to \( F \), a generalized Kannan - Reich type mapping. Let \( X_n \) and \( X_0 \) be fixed points of \( F_n \) and \( F \) resp. Then \( X_n \to X_0 \).

The following is a simple consequences of the definition of Hausdorff metric \( H \). Let \( A, B \in CB(X) \) and \( a \in A \). if \( \eta > 0 \), then there exist \( b \in B \) such that \( d(a, b) \leq H(A, B) + \eta \), i.e., \( d(a, b) \leq PH(A, B) \) where \( P > 1 \). If \( A, B, a \in C(X) \) and \( a \in A \), then there exist \( b \in B \) such that \( d(a, b) \leq H(A, B) \).

The following theorem were proved by Bose and Mukherjee².

**Theorem 5**

Let \((X, d)\) be complete bounded metric space and let \( F : X \to CL(X) \) be mapping satisfying the following condition.

\[
d(F(x), F(y)) \leq \alpha \left( [d(x, F(x))]^2 + [d(y, F(y))]^2 \right) + \beta d(x, y)
\]

where \( \alpha \) and \( \beta \) are non-negative numbers \( 0 < 2 \alpha + \beta < 1 \). then \( F \) has a fixed point.

**Proof**

Let \( x_0 \in X \) consider the sequence \( \{x_n\} \) where \( x_{n+1} \in F(x_n) \)

1. Let us assume \( d(F(x_n), F(x_1)) = 0 \) \( \Rightarrow \) then \( F \) has fixed point.

II. Now if \( d(F(x_n), F(x_1)) \neq 0 \) Then E as number \( h > d(F(x_n), F(x_1)) \).

Let \( h = p^{-1} d(F(x_n), F(x_1)) \) Where \( P = (a+b) \)

Then

\[
d(x_n, x_1) \leq p \left( \alpha \left( [d(x_n, F(x_n))]^2 + [d(x_1, F(x_1))]^2 \right) + \beta d(x_n, x_1) \right)
\]

\[
\rho d(x_n, x_1) \leq \alpha \left( [d(x_n, x_1)]^2 + [d(x_1, x_1)]^2 \right) + \beta d(x_n, x_1)
\]
< α[d(x₀, x₁) + d(x₁, x₂)] + βd(x₀, x₁) \\
< α[d(x₀, x₁) + d(x₁, x₂)] + βd(x₀, x₁) \\
Pd(x₁, x₂) < α(α + β)d(x₀, x₁) + αd(x₁, x₂) \\
d(x₁, x₂) < \left(\frac{α + β}{p - α}\right)d(x₀, x₁) \\

continuing in similar fashion there exist xₙ ∈ F(xₙ) such that \\
d(x₂, x₃) < \left(\frac{α + β}{p - α}\right)^2d(x₀, x₁) \\

We have α > 0 α = \left(\frac{α + β}{p - α}\right) \\

Further, \\
d(x₂₋₁, x₂₋₂) < \left(\frac{α + β}{p - α}\right)^{2n-1}d(x₀, x₁) \\

It can be easily seen that the sequence is Cauchy sequence and hence converges to some points u ∈ X consider. \\
d(F(u), u) ≤ d(F(u), xₙ₋₁) + d(xₙ₋₁, u) \\
≤ d(F(u), F(xₙ)) + d(xₙ₋₁, u) \\
≤ p⁻¹\left[\frac{α [d(u, F(u))]² + (xₙ, F(xₙ))}{d(u, F(u)) + d(xₙ, F(xₙ))}\right] + βd(u, xₙ) \\
Pd(F(u), u) < α [d(u, F(u)) + d(xₙ₋₁,f(xₙ))] + β d(u,xₙ₋₁) + d(xₙ₋₁, u) \\
⇒ (P-α) d(F(u), u) ≤ αd(xₙ₋₁, u) + β d (u, xₙ) + d (xₙ₋₁,u) \\
Taking limit n →∞ then \\
(P-α) d (F(u), u) = 0 \\
⇒ F (u) = u \\

u is a fixed point for F. 

REFERENCES

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